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Generalized Bernstein Type Inequalities for the Higher Derivatives of a Complex Polynomial

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Abstract:

Let** $v(z)$ **be a polynomial of degree** m **having no zero zero in** $|z| \le a$, $a \ge 1$, **then for** $1 \le T \le a$, **Dewan and Bidkham [J. Math. Anal. Appl., vol. 166, pp. 319-324, 1992] proved

$$\max_{|z|=T} |v'(z)| \le n \frac{(T+a)^{m-1}}{(1+a)^m} \max_{|z|=1} |v(z)|.$$

The result is best possible and extremal polynomial is $v(z) = (z+a)^m$.

In this paper, by involving certain co-efficients of the polynomial $v(z)$, we prove a result concerning the estimate of maximum modulus of higher derivatives of $v(z)$, which not only improves as well as generalizes the above result, but also has interesting consequences as special cases.

1. INTRODUCTION

It was for the first time, Bernstein [9, 10] investigated an upper bound for the maximum modulus of the first derivative of a complex polynomial on the unit circle in terms the maximum modulus of the polynomial on the same circle and proved the following famous result known as Bernstein's inequality that if $v(z)$ is a polynomial of degree m, then

$$\max_{|z|=1} |v'(z)| \le m \max_{|z|=1} |v(z)|. \tag{1.1}$$

Inequality (1.1) is best possible and equality occurs for $v(z) = \lambda z^m$, $\lambda \ne 0$, is any complex number.

If we restrict to the class of polynomials having no zero in $|z| < 1$, then inequality (1.1) can be sharpened as

$$\max_{|z|=1} |v'(z)| \le \frac{m}{2} \max_{|z|=1} |v(z)|. \tag{1.2}$$

The result is sharp and equality holds in (1.2) for $v(z) = \alpha + \beta z^m$, where $|\alpha| = |\beta|$.

Inequality (1.2) was conjectured by Erdös and later proved by Lax [7].

Simple proofs of this theorem were later given by de-Bruijn [3], and Aziz and Mohammad [1].

It was asked by Professor R.P. Boas that if $v(z)$ is a polynomial of degree m not vanishing in $|z| < a$, $a > 0$, then how large can

$$\left\{ \max_{|z|=1} |v'(z)| \Big/ \max_{|z|=1} |v(z)| \right\} \text{ be?} \tag{1.3}$$

A partial answer to this problem was given by Malik [8], who proved

Theorem A. If $v(z)$ is a polynomial of degree m having no zero in the disc $|z| < a$, $a \ge 1$, then

$$\max_{|z|=1} |v'(z)| \le \frac{m}{1+a} \max_{|z|=1} |v(z)|. \tag{1.4}$$

The result is best possible and equality holds for $v(z) = (z+a)^m$.

For the class of polynomials not vanishing in $|z| < a$, $a \le 1$, the precise estimate for maximum of $|v'(z)|$ on $|z| = 1$, in general, does not seem to be easily obtainable.For quite some time, it was believed that if $v(z) \ne 0$ in $|z| < a$, $a \le 1$, then the inequality analogous to (1.4) should be

$$\max_{|z|=1} |v(z)| \le \frac{m}{1+a^m} \max_{|z|=1} |v(z)|. \tag{1.5}$$

till Professor E.B. Saff gave the example $v(z)=\left(z-\frac{1}{2}\right)\left(z+\frac{1}{3}\right)$ to counter this belief.

Dewan and Bidkham [4] generalized Theorem A by considering any circle that lies in a closed circular annulus of radii 1 and a where $a \geq 1$. In fact, they prove

Theorem B. *If* $v(z)$ *is a polynomial of degree* m *having no zero in* $|z|<a$, $a\geq 1$, *then for* $1\leq T\leq a$,

$$\max_{|z|=T}|v'(z)| \leq m\frac{(T+a)^{m-1}}{(1+a)^{m}}\max_{|z|=1}|v(z)|. \tag{1.6}$$

The result is best possible and extremal polynomial is $v(z)=(z+a)^m$.

In this paper, by involving some co-efficients of the polynomial $v(z)$ and also $\min_{|z|=a}|v(z)|$, we obtain a result which is an improvement and a generalization of (1.6) by further extending for the sth derivative of $v(z)$ and maxima are considered on two different circles lying both inside and on any circle . More precisely, we have

Theorem. *If* $v(z)=\sum_{\upsilon=0}^{m}c_\upsilon z^\upsilon$ *is a polynomial of degree* m *having no zero in* $|z|<a$, $a>0$, *then for* $0<t\leq T\leq a$, *and* $1\leq s\leq m$,

$$\max_{|z|=T}\left|v^{(s)}(z)\right| \leq \frac{m(m-1).......(m-s+1)}{T^s+\delta_{a,s}} \times\left[\left(\frac{T+a}{t+a}\right)^m\max_{|z|=t}|v(z)|-\left\{\left(\frac{T+a}{t+a}\right)^m-1\right\}\min_{|z|=a}|v(z)|\right] \tag{1.7}$$

where

$$\delta_{a,s}=\frac{C(m,s)|c_0|a^{s+1}+|c_s|Ta^{2s}}{C(m,s)|c_0|T+|c_s|a^{s+1}} \textit{ with } C(m,s)=\frac{m!}{s!(m-s)!}.$$

The result is best possible for $s=1$ *and equality in* (1.7) *holds for* $v(z)=(z+a)^m$.

Remark 1.1. Putting $t=1$, our theorem gives the following result, which is an improvement of Theorem B.

Corollary 1.1. *If* $v(z)=\sum_{\upsilon=0}^{m}c_\upsilon z^\upsilon$ *is a polynomial of degree* m *having no zero in* $|z|<a$, $a\geq 1$, *then for* $1\leq T\leq a$, *and* $1\leq s\leq m$,

$$\max_{|z|=T}\left|v^{(s)}(z)\right| \leq \frac{m(m-1).......(m-s+1)}{T^s+\delta_{a,s}} \times\left[\left(\frac{T+a}{1+a}\right)^m\max_{|z|=t}|v(z)|-\left\{\left(\frac{T+a}{1+a}\right)^m-1\right\}\min_{|z|=a}|v(z)|\right] \tag{1.8}$$

where $\delta_{a,s}$ *is as defined in the theorem.*

Remark 1.2. Corollary 1.1 provides a generalization of a result due to Aziz and Rather [2].

Remark 1.3. If we assign $T=s=1$ in Corollary 1.1, it reduces to a result of Govil et.al [6].

Corollary 1.2. *If* $v(z)=\sum_{\upsilon=0}^{n}c_\upsilon z^\upsilon$ *is a polynomial of degree* m *having no zero in* $|z|<a$, $a\geq 1$, *then*

$$\max_{|z|=1}|v'(z)| \leq \frac{|c_1|a^2+m|c_0|}{(1+a^2)m|c_0|+2|c_1|a^2}\max_{|z|=1}|v(z)|. \tag{1.9}$$

Remark 1.4. Corollary 1.2 is an improved version of a well-known inequality proved by Malik [8] under the same set of hypotheses.

Remark 1.5. If we put $s=t=T=a=1$, our theorem gives inequality (1.2), conjectured by Erdös and later proved by Lax [7].

2. LEMMA

The following lemmas are needed for the proof of the theorem.

Lemma 2.1. *If* $v(z)=\sum_{\upsilon=0}^{m}c_\upsilon z^\upsilon$ *is a polynomial of degree* n *such that* $v(z)\neq 0$ *for* $|z|<a$, $a\geq 1$, *then*

$$\mu_{a,s}\left|v^{(s)}(z)\right| \le \left|u^{(s)}(z)\right| \text{ for } |z|=1, \tag{2.1}$$

and

$$\frac{1}{C(m,s)}\frac{|c_s|}{|c_0|}a^s \le 1,$$

where

$$u(z)=z^m\overline{v\left(\frac{1}{\bar{z}}\right)},$$

$$\mu_{a,s}=\frac{C(m,s)|c_0|a^{s+1}+|c_s|a^{2s}}{C(m,s)|c_0|+|c_s|a^{s+1}} \text{ and}$$

$C(m,s)$ is as defined in our theorem.

The above result is due to Aziz and Rather [2].

Lemma 2.2. *If $v(z)=c_0+\sum_{\upsilon=\mu}^{m}c_\upsilon z^\upsilon$, $1\le\mu\le m$, is a polynomial of degree m such that $v(z)\ne 0$ in $|z|<a$, $a>0$, then for $0<t\le T\le a$,*

$$\max_{|z|=T}|v(z)| \le A\max_{|z|=t}|v(z)|-(A-1)\min_{|z|=a}|v(z)|. \tag{2.2}$$

where

$$A=\left(\frac{T^\mu+a^\mu}{t^\mu+a^\mu}\right)^{\frac{m}{\mu}}.$$

Equality holds in (2.2) *for $v(z)=\left(z^\mu+a^\mu\right)^{\frac{m}{\mu}}$ where m is a multiple of μ.*

This Lemma is due to Dewan et. al [5].

3. PROOF OF THE THEOREM

If $v(z)$ has no zero in $|z|<a$, $a>0$ and if $0<t\le T\le a$, then $V(z)=v(Tz)$ has no zero in $|z|<\frac{a}{T}$, $\frac{a}{T}\ge 1$. Thus, applying Lemma 2.1 to $P(z)$, we get for $1\le s\le n$ and $|z|=1$,

$$\delta'_{a,s}\left|V^{(s)}(z)\right| \le \left|U^{(s)}(z)\right|, \tag{3.1}$$

where

$$\delta'_{a,s}=\frac{1}{T^s}\left\{\frac{C(m,s)|c_0|a^{s+1}+|c_s|T\,a^{2s}}{C(m,s)|c_0|T+|c_s|a^{s+1}}\right\},$$

and

$$u(z)=z^m\overline{v\left(\frac{1}{\bar{z}}\right)}.$$

Now if $f(z)$ is a polynomial of degree n having all its zeros in $|z|\le 1$, then $g(z)=z^m\overline{v\left(\frac{1}{\bar{z}}\right)}$ has no zero in $|z|<1$.Hence using inequality (2.1) of Lemma 2.1 with $a=1$, we have for $|z|=1$

$$\left|g^{(s)}(z)\right| \le \left|f^{(s)}(z)\right|. \tag{3.2}$$

Let $M=\max_{|z|=1}|V(z)|$, then for every real or complex number we have for γ with $|\gamma|>1$, it follows by Rouche's theorem that the polynomial $R(z)=V(z)-\gamma M z^m$ has all its zeros in $|z|<1$. Suppose

$$S(z)=z^m\overline{R\left(\frac{1}{\bar{z}}\right)}=z^m\overline{V\left(\frac{1}{\bar{z}}\right)}-\bar{\gamma}M=U(z)-\bar{\gamma}M.$$

Applying (3.2) to $R(z)$, we get for $1\le s\le n$ and $|z|=1$

$$\left|S^{(s)}(z)\right| \le \left|R^{(s)}(z)\right|,$$

which implies for $|z|=1$

$$\left|U^{(s)}(z)\right| \le \left|V^{(s)}(z)-\gamma M\, m(m-1).......(m-s+1)z^{m-s}\right|. \tag{3.3}$$

Since $V(z)$ is of degree n, it is evident that the polynomial $V^{(s)}(z)$ is of degree $(n-s)$ and repeated application of Bernstein's inequality (1.1) to $V(z)$ yields for $|z|=1$,

$$\left|V^{(s)}(z)\right| \le M\, m(m-1).......(m-s+1). \tag{3.4}$$

Further, for suitable choice of the argument of γ in (3.3), we have for $|z|=1$,

$$\left|U^{(s)}(z)\right| \le \left||\gamma|M\, m(m-1).......(m-s+1)-\left|V^{(s)}(z)\right|\right|,$$

which become by making limit as $|\gamma|\to 1$

$$\left|U^{(s)}(z)\right| \le \left|M\, m(m-1).......(m-s+1) - \left|V^{(s)}(z)\right|\right|$$

$$= M\, m(m-1).......(m-s+1) - \left|V^{(s)}(z)\right|. \quad \text{[Using (3.4)]}$$

That is for $|z|=1$,

$$\left|V^{(s)}(z)\right| + \left|U^{(s)}(z)\right| \le m(m-1).......(m-s+1)M\,. \qquad (3.5)$$

Inequality (3.5) in conjunction with inequality (3.1) gives for $|z|=1$

$$\left|V^{(s)}(z)\right| \le \frac{1}{1+\delta'_{a,s}} m(m-1).......(m-s+1) \max_{|z|=1} |V(z)|\,. \qquad (3.6)$$

Replacing $V(z)$ by $v(Rz)$, we have

$$T^s \left|v^{(s)}(z)\right| \le \frac{1}{1+\delta'_{a,s}} m(m-1).......(m-s+1) \max_{|z|=T} |v(z)|\,.$$

which is equivalent to

$$\left|v^{(s)}(z)\right| \le \frac{1}{T^s+\delta_{a,s}} m(m-1).......(m-s+1) \max_{|z|=T} |v(z)|\,, \qquad (3.7)$$

where $\delta_{a,s}$ is as defined in the theorem.

When $\mu=1$, Lemma 2.2 becomes for $0<t\le T\le a$

$$\max_{|z|=T} |v(z)| \le B \max_{|z|=t} |v(z)| - (B-1) \min_{|z|=a} |v(z)|\,. \qquad (3.8)$$

where

$$B = \left(\frac{T+a}{t+a}\right)^m.$$

Using inequality (3.8) to inequality (3.7), we obtain the required inequality, and this completes the proof of the theorem.

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